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–Brief Paper–

EXPONENTIAL STABILIZATION OF SWITCHED DISCRETE-TIME SYSTEMS WITH ALL UNSTABLE MODES

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ABSTRACT

This paper studies the exponential stabilization of switched discrete-time systems whose subsystems are unstable. A new sufficient condition for the exponential stability of the class of systems is proposed. The result obtained is based on the determination of a lower bound of the maximum dwell time by virtue of the multiple Lyapunov functions method. The key feature is that the given stability condition does not need the value of the Lyapunov function to uniformly decrease at every switching instant. An example is provided to illustrate the effectiveness of the proposed result.

Key Words: Switched discrete-time systems, Stability, Dwell time, Multiple Lyapunov functions.

I. INTRODUCTION

Switched systems have attracted a great deal of attention recently not only because of their inherent complexity, but also because they have numerous applications in traffic control systems, power systems, etc. Stability is a fundamental problem for switched systems, which has been extensively investigated and fruitful results have been reported [1–11]. However, most results focus on switched continuous-time systems. It is well acknowledged that, in practice, more and more complicated systems need to be modeled in discrete-time. Thus, it is necessary to investigate the stability of switched discrete-time systems. Unstable subsystems may drive the state to infinity, and may also be detrimental to system performance. Moreover, unstable subsystems may inevitably appear and, in some situations, cannot be avoided. Thus, designing switching laws to stabilize a switched system whose subsystems are all unstable is one of the most serious challenges for switched systems [2,4,8,12]. This problem has been widely studied [13–17], and most designs are state-dependent switching strategies such as the min-projection strategy and largest region function approach. Very few results focus on time-dependent switching laws, which motivates us to continue to design a time-dependent switching law for easy implementation.

For time-dependent switching signals, almost all existing results on switched systems assume that all subsystems or partial subsystems are stable [6,18–20], but it should be pointed out that these results require the existence of (at least one) stable subsystem to guarantee the stability of the switched system. The main idea of these results is to make the stable subsystems run sufficiently long enough to compensate the state divergence made by unstable systems. Obviously, when all the subsystems are unstable, this idea is not applicable, because there exists no stable interval to compensate for the state divergence effect. Since the above idea is not suitable for cases in which all subsystems are unstable, we have to find other ways to establish stability. For switched continuous-time systems, the stability problem of switched systems with all unstable subsystems was investigated in [21] by using the discretized Lyapunov function technique. But the condition in [21] is conservative and is often difficult to check. Switched nonlinear systems were converted into interconnected switched nonlinear systems so that the small-gain theorem was used to establish the stability condition of the switched system under the periodical stabilizing switching law [19]. To the best of our knowledge, there have been no results in the literature that consider the stability issue of switched discrete-time systems with all unstable subsystems.

In view of the importance of this problem partially inspired by [21], in this paper we study the stability of switched discrete-time systems with all unstable subsystems. A new condition for the exponential stability of switched discrete-time systems whose subsystems are all unstable is given. Unlike [21], we remove the sufficient condition of [21], which is usually difficult to check. The main features of this paper are as follows. Compared to [22], the proposed method does not require exponential stability of each sub-state switched system and the switching can be aperiodic. A new sufficient condition, which is easy to verify for the exponential stability of the system with all unstable modes, is provided and a lower bound of the maximum dwell time preserving stability is given. The proposed method also does not require the value of the Lyapunov function to uniformly decrease at every switching instant under certain mild conditions.

II. SYSTEM DESCRIPTION AND PRELIMINARIES

Consider a class of switched discrete-time systems

$$x(k+1) = f_{\sigma(k)}(x(k)), \tag{1}$$

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where $x(k)\in R^n$ is the state vector, $\sigma(k):[0,\infty)\to M$ is the switching signal, *i.e.* $\sigma(k)=i_l\in M$ for $k\in[k_l,k_{l+1})$, where k_l is the lth switching time instant, $M=\{1,2,\cdots,m\}$, $l=0,1,2,\cdots$. f_i: $R^n\to R^n$ are smooth functions with $f_i(0)=0$, $\forall i\in M$. The switching signal can be characterized by a switching sequence

$$\Sigma=\{x_0;(i_0,k_0),(i_1,k_1),\cdots,(i_l,k_l),\cdots|i_l\in M,l=0,1,2,\cdots\},\tag{2}$$

where t_0 is the initial time, and x_0 is the initial state. When $k\in[k_l,k_{l+1})$, $\sigma(k)=i_l$, *i.e.* the i_l subsystem is active. Since we consider unstable subsystems, without loss of generality, the origin is not a stable (attractive) equilibrium for any mode.

Definition 1[6]. A switching signal σ is said to have a dwell time τ if $k_{l+1}-k_l\geq\tau\geq1$, $l=0,1,2,\cdots$, where k_l and k_{l+1} are successive switching instants.

Definition 2. Let $N_\sigma(t_s,\ t_u)$ denote the number of discontinuities of a switching signal σ on the interval $(t_s,\ t_u)$. We say that σ has a maximum dwell time τ if

$$N_\sigma(t_s,\ t_u)>\frac{t_u-t_s}{\tau}\tag{3}$$

It is well known that unstable subsystems often result in instability for switched systems. Usually, when stable subsystems and unstable subsystems co-exist, we can use the stable subsystems to compensate for the effect of the unstable subsystems. However, this idea is not effective when the subsystems are all unstable. In this paper, by constructing decreasing Lyapunov functions at the switching instants, and utilizing the maximum dwell time technique, exponential stabilization of the switched system composed of only unstable subsystems is obtained.

III. STABILITY ANALYSIS

In this section, we provide a condition for the exponential stability of switched discrete-time system (1) with all subsystems unstable.

Theorem 1. Consider switched system (1). Let $\alpha>1$ and $0<\mu<1$ be given constants. Suppose that there exist continuous positive function $V_{\sigma(k)}(x(k))$: $R^n\to R$, and positive numbers κ_1 and κ_2 such that

$$\kappa_1\|x\|^2\leq V_i(x(k))\leq\kappa_2\|x\|^2,\ \ \forall i\in M\tag{4}$$

$$V_i(x(k+1))\leq\alpha V_i(x(k)),\ \ k\in[k_l,\ \ k_{l+1}),\ \ \alpha>1,\ \ \forall i\in M\tag{5}$$

$$V_{\sigma(k_l)}(x(k_l))\leq\mu V_{\sigma(k_{l-1})}(x(k_l)),\ \ 0<\mu<1.\tag{6}$$

Then system (1) is exponentially stable under any switching signal satisfying

$$\frac{1}{\tau}\geq-\frac{\ln\alpha}{\ln\mu},\tag{7}$$

where $k_{l+1}-k_l=\tau_l\geq\tau\geq1$, $l=0,1,2,\cdots$.

Proof. When $\forall k\in[k_l,\ \ k_{l+1})$, for $\sigma(k)=i\in M$, the ith subsystem of switched system (1) is active. From (5) it holds that

$$V_{\sigma(k)}(x(k))\ \leq\alpha^{k-k_l}\ \ V_{\sigma(k_l)}(x(k_l))\leq\mu\alpha^{k-k_l}V_{\sigma(k_{l-1})}(x(k_l)).\tag{8}$$

Then, by (6), one obtains

$$\begin{aligned}V_{\sigma(k)}(x(k))\ &\leq\alpha^{k-k_l}\ \ V_{\sigma(k_l)}(x(k_l))\leq\mu\alpha^{k-k_l}V_{\sigma(k_{l-1})}(x(k_l))\\ \leq\mu\alpha^{k-k_l}\alpha^{k_l-k_{l-1}}V_{\sigma(k_{l-1})}(x(k_{l-1}))\ &\leq\mu^2\alpha^{k-k_l}\alpha^{k_l-k_{l-1}}V_{\sigma(k_{l-2})}(x(k_{l-1}))\\ &\leq\cdots\\ \leq\mu^{N_\sigma(k_0,k)}\alpha^{k-k_0}V_{\sigma(k_0)}(x(k_0))\ &\leq\mu^{\frac{k-k_0}{\tau}}\alpha^{k-k_0}V_{\sigma(k_0)}(x(k_0))\end{aligned}\tag{9}$$

Since the dwell time satisfies (7), one can readily obtain

$$\alpha\mu^{\frac{1}{\tau}}\leq\alpha\mu^{-\frac{\ln\alpha}{\ln\mu}}=\alpha\mu^{-\log_\mu^\alpha}=\alpha\alpha^{-1}=1.\tag{10}$$

Denote $\beta=\sqrt{\alpha\mu^{\frac{1}{\tau}}}$. Then, the system state satisfies

$$\|x(k)\|^2\leq\frac{1}{\kappa_1}V_{\sigma(k)}(x(k))\leq\frac{\kappa_2}{\kappa_1}\beta^{2(k-k_0)}\|x(k_0)\|^2$$

which means $\|x(k)\|\leq\sqrt{\frac{\kappa_2}{\kappa_1}}\beta^{(k-k_0)}\|x(k_0)\|.$ (11)

Therefore, system (1) is globally exponentially stable, which completes the proof.

However, in many actual applications, since the sequence $\{k_0,k_1,k_2,\cdots,k_l,\cdots\}$ cannot be specified in advance, it is difficult to check condition 6 for all switching instants k_l as $l\to\infty$. Thus, Theorem 1 may be impractical in some cases. To overcome this difficulty, we convert Theorem 1 for the linear case.

IV. STABILIZATION FOR SWITCHED DISCRETE-TIME LINEAR SYSTEMS

Consider a switched discrete-time linear system in the form of

$$x(k+1)=A_{\sigma(k)}x(k),\tag{12}$$

where $x(k)$ and $\sigma(k)$ are described as in (1). Without loss of generality, A_i has at least one eigenvalue with modulus larger than 1, $i\in M$.

A stabilization condition of system (12) can be given in the following theorem.

Theorem 2. Consider the switched discrete-time linear system (12). For given scalars $\alpha>1$ and $0<\mu<1$ if there exist matrices $P_i>0$ such that the following inequalities hold

$$\begin{bmatrix}-\alpha P_i & A_i^TP_i\\ P_iA_i & -P_i\end{bmatrix}<0,\ \ \alpha>1,\ \ \forall\ i\in M,\tag{13}$$

$$A_i^{T\tau}P_jA_i^\tau-\mu P_i<0,\ \ 0<\mu<1,\ \ \forall i\neq j\in M,\tag{14}$$

then, system (12) is exponentially stable under any switching signal satisfying

$$\frac{1}{\tau} \geq -\frac{\ln\alpha}{\ln\mu}. \tag{15}$$

Proof. Construct the following Lyapunov function for system (12)

$$V_{\sigma(k)}(x(k)) = x^T(k)P_{\sigma(k)}x(k). \tag{16}$$

Obviously, (4) is satisfied.

It is seen that, for all $k \in [k_l, k_{l+1})$, $\sigma(k) = i \in M$, the Lyapunov function (16) along an arbitrary trajectory of (12) satisfies

$$\begin{aligned} V_i(x(k+1)) - \alpha V_i(x(k)) &= x^T(k+1)P_i x(k+1) - \alpha x^T(k)P_i x(k) \\ &= x^T(k)\left(A_i^T P_i A_i - \alpha P_i\right)x(k). \end{aligned} \tag{17}$$

By the Schur Compliment Lemma and from (13), we have $V_i(x(k+1)) < \alpha V_i(x(k))$. Thus, condition 5 in Theorem 1 holds.

At the switching times $k = k_{l+1}$, the switching control jumps to $\sigma(k) = j \in M$. Using inequalities (14) we have

$$\begin{aligned} &V(x(k_{l+1})) = x^T(k_{l+1})P_j x(k_{l+1}) = x^T(k_l)A_i^{T\tau_l}P_j A_i^{\tau_l}x(k_l) \\ &< \mu x^T(k_l)A_i^{T(\tau_l-\tau)}P_i A_i^{(\tau_l-\tau)}x(k_l) < \mu x^T(k_l)P_i x(k_l) \\ &= \mu V(x(k_l)) \leq \mu V(x(k_{l+1})). \end{aligned} \tag{18}$$

Obviously, $V_{\sigma(k_{l+1})}(x(k_{l+1})) < \mu V_{\sigma(k_l)}(x(k_l)) \leq \mu V_{\sigma(k_l)}(x(k_{l+1}))$, and (6) in Theorem 1 is satisfied. Therefore, the exponential stability of system (12) governed by any switching law $\sigma(k)$ satisfies (15) and can be obtained based on Theorem 1.

Remark 1. For switched system (12), Theorem 2 indicates that no stability for any subsystem is required. This can be seen from condition (13).

Remark 2. It is worth noting that the conditions of Theorem 2 do not include a switching signal. Thus, these conditions can be easily checked.

V. EXAMPLE

Consider the switched discrete-time linear system (12) with two subsystems as follows

$$A_1 = \begin{bmatrix} -0.15 & -0.1 \\ -0.01 & -1.12 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1.15 & -0.4 \\ -0.1 & 0.17 \end{bmatrix}.$$

The eigenvalues of A_1 are $\lambda_{11} = -0.1490$ and $\lambda_{12} = -1.1210$, and eigenvalues of A_2 are $\lambda_{21} = 1.1892$ and $\lambda_{22} = 0.1308$. Obviously, they are unstable because both have eigenvalues larger than 1.

Choose the initial state as $x(0) = [-0.5\ \ 0.5]^T$. Given the scalars, $\alpha = 2$ and $\mu = 0.1$. Therefore, we can calculate the lower bound of the maximum dwell time as $1 \leq \tau_D \leq 3.3219$. Then, by (13) and (14), if we select $\tau_D = 2$, the following feasible solutions can be obtained

$$P_1 = \begin{bmatrix} 0.2684 & 0.8125 \\ 0.8125 & 20.6316 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 9.5023 & -2.3478 \\ -2.3478 & 1.3615 \end{bmatrix}.$$

Fig. 1 illustrates the state response. Obviously, the switched system is stabilized by the switching law, which satisfies (15). The corresponding value of the Lyapunov function is shown in Fig. 2, in which we can see that the value of the Lyapunov function may increase during the time (*e.g.* in [0,2]), but the switching behaviors compensate for the increments (*e.g.* the Lyapunov function decreases at the switching point k_1) and the value of the Lyapunov function finally converges to zero. Fig. 3 is the corresponding switching signal.

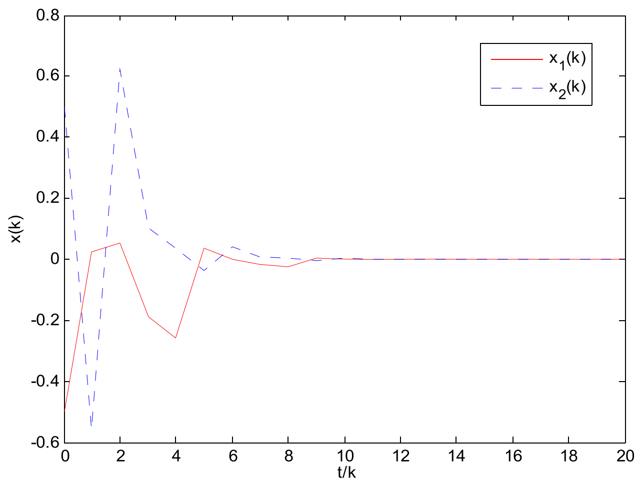

Fig. 1. State trajectories of $x(k)$. [Color figure can be viewed at wileyonlinelibrary.com]

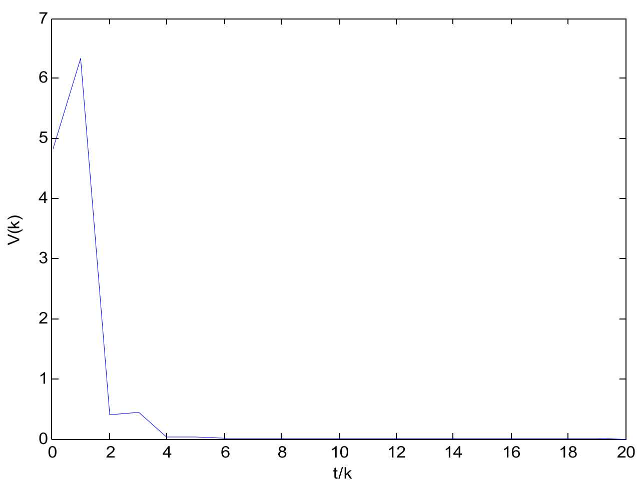

Fig. 2. Evolution of Lyapunov function. [Color figure can be viewed at wileyonlinelibrary.com]

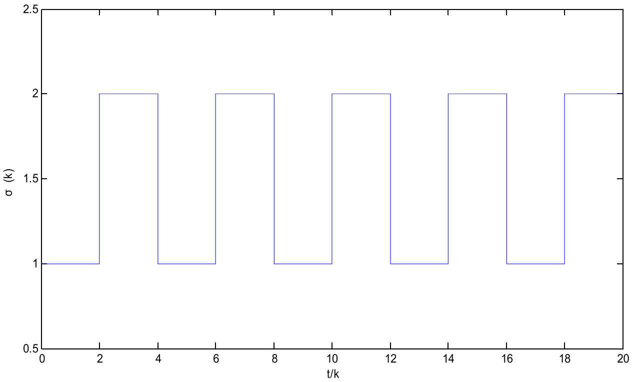

Fig. 3. Switching signal. [Color figure can be viewed at wileyonlinelibrary.com]

VI. CONCLUSIONS

In this paper, the stability problem for switched discrete-time systems with all unstable subsystems has been addressed. By constructing a class of decreasing Lyapunov functions, the exponential stability of the switched systems is guaranteed by the maximum dwell time approach. The key feature of the dwell time calculation is that the proposed stability condition does not require the Lyapunov function to uniformly decrease at each switching instant. There are some relevant problems that deserve further study. For example, the approach of this paper can be used to improve some of the previous results in [23] and a proper switching law can be designed to relax those restrictions.

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