Systems & Control Letters 156 (2021) 105012

Contents lists available at ScienceDirect

Systems & Control Letters

journal homepage: www.elsevier.com/locate/sysconle

Hybrid adaptive control of nonlinear systems with non-Lipschitz nonlinearities

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ARTICLE INFO

Article history:
Received 21 April 2021
Received in revised form 10 June 2021
Accepted 29 July 2021
Available online 17 August 2021

Keywords:
Hybrid adaptive control
Non-Lipschitz nonlinearities
Multiple models
Artificial neural networks

ABSTRACT

In this paper, a hybrid adaptive control method is proposed for a class of discrete-time multiple-inputs-multiple-outputs (MIMO) systems with *non-Lipschitz* nonlinearities by introducing a novel Riemann sum operator with variable step size. Firstly, for the first order approximation of the original nonlinear system, a linear adaptive controller is developed to assure essential boundedness of all the signals but with unsatisfactory performance. Secondly, an artificial neural network (ANN)-based nonlinear adaptive controller is designed for the original system to improve response performance, nevertheless, may lead to instability. Thirdly, switching mechanism between the linear adaptive controller and the nonlinear one is suitably designed to achieve stability and improved output tracking performance simultaneously. Finally, a numerical example is provided to verify the effectiveness of the proposed method.

1. Introduction

The past over half a century has witnessed the development of adaptive control [1–4]. Different criteria bring different categories of adaptive control. Although under each of the categories there are many research articles and a few specialized monographs addressing the fundamentals and advanced developments of adaptive control, as pointed out by Tao [3], multivariable adaptive control of multiple-inputs-multiple-outputs (MIMO) nonlinear systems is one of open areas in adaptive control, which is the focus of this paper.

To the considered multivariable adaptive control of MIMO nonlinear systems, the most relevant results are work by [5] and our previous results [6,7]. In [5], the authors for the first time established adaptive control of a class of nonlinear discrete time dynamical systems with boundedness of all signals by using a linear robust adaptive controller and a neural networks based nonlinear adaptive controller, and switching between them by a suitably defined switching law. However, [5] assumed global boundedness of nonlinearities and thus restricts the applicability range of the considered systems. [6,7] extended the result of [5] by relaxing the assumption of global boundedness on the higher-order nonlinearities to allow the nonlinearities to be Lipschitz and linear growth, respectively. However, the satisfaction of such assumptions are usually difficult to justify in practical engineering. Moreover, non-Lipschitz nonlinearities widely exist in real-world systems that are intuitively those nonlinearities whose first derivative does not exist. In real world, there are some practically important systems contain non-Lipschitz nonlinearities which cannot be handled with the tools developed for the systems with Lipschitz nonlinearities. For example, impulsive disturbances in wireless networked control systems [8] and in the reverberant system of the Virginia Tech plate testbed [9], the fluctuations of current in the deterministic electric network in [10], and the nonlinear force in the spin systems of anharmonic oscillators in [11]. Although some simple non-Lipschitz functions can be approximated by some known functions, like the Heaviside distribution can be approximated by the logistic function, we herein attempt to directly deal with the non-Lipschitz nonlinearities satisfying certain conditions. In these cases, the above controller will result in instability. To extend these methods to handle the non-Lipschitz nonlinearities, new operator or system transformation with novel controllers need to be designed as well as the stability and performance assessment theorem. To our best knowledge, there have not been results providing solution to the above challenges, thus this article aims to achieve both stability and improved performance of multivariable adaptive control of MIMO nonlinear systems with *non-Lipschitz nonlinearities*.

Considering only the capability of ensuring stability and performance, some classical robust control schemes such as sliding mode control, robust Lyapunov control and mixed flatness and model-free control may work as well. Sliding mode control is a variable structure control method, which changes the dynamics of a nonlinear system by application of a discontinuous control

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https://doi.org/10.1016/j.sysconle.2021.105012
0167-6911/

signal that forces the system to "slide" along a manifold (i.e., the sliding surface), that remarkably rejects certain disturbance and parameter variations. For example, [12] proposed a novel sliding mode control method for stochastic systems under the event-triggering mechanism to achieve both stabilization and better performance. Robust Lyapunov control is to design a control input such that an uncertain nonlinear system is of Lyapunov stability. However, the conventional sliding mode control and robust Lyapunov control are model-based although they have relatively low requirement of model precision. Mixed flatness based and model free control is an ingenious combination of flatness-based control and model free control to cope with control of uncertain nonlinear systems, where partially known part of the system happens to be flat and the unknown part describing unknown *bounded* dynamics can be compensated for or estimated online by model free control such as intelligent proportion integration differentiation (PID) method in [13]. The mixed flatness and model-free control does not require detailed model, nevertheless, it can be viewed as another way of predictive control but without optimizing certain performance. Recently, as the prevalence of data-driven and machine learning, learning-based model-free control approaches have been widely studied. In [14], a data-driven control method for single-input-single-output (SISO) systems with unknown control gain was proposed based on the idea in [13]. Its main idea is to recursively learn the input–output mapping at each state relying on a feedback linearizability assumption. Recently, a singularity-free fixed-time fuzzy control for robotic systems with the position tracking error constraint is proposed in [15]. In [16], the event-driven model-free control was applied to two nonlinear MIMO motion models, and the control performance such as tracking error, algorithm complexity and robustness are compared with sliding mode, backstepping and PID control. In conclusion, classical robust control approaches ensure the performance only in a qualitative manner [12,13], whereas the methods proposed by [14,16] and this paper can quantify the stability results and tracking performance, respectively.

To achieve model-free adaptive control for MIMO systems with optimal tracking performance as well as handling *non-Lipschitz nonlinearities*, in this paper, we propose a hybrid adaptive control strategy by introducing a new Riemann sum operator with variable step size. Firstly, a linear adaptive controller is developed for its linearized estimate model to guarantee *essential* boundedness of all the signals with probably unsatisfactory performance. The *essential* boundedness is defined as the signals are bounded *almost everywhere* except for the finite points that unbounded perturbation occurs. Compared to the conventional BIBO stability, essential boundedness is more practical for real-world engineering. Then, a data-driven nonlinear adaptive controller based on ANN is designed to improve the performance which is measured by the tracking error between system output and reference trajectory. But it is possible to lead to instability. Finally, a switching mechanism between the two controllers is properly designed to achieve both essential stability and improved tracking performance simultaneously.

Compared to the relevant existing results in the literature, the main features of this paper can be concluded in twofolds: (1) A novel Riemann sum operator with variable step size is introduced for the first time to multiple models adaptive control of MIMO nonlinear systems to deal with the non-Lipschitz nonlinearity; (2) The proposed method can cope with *non-Lipschitz nonlinearities* in an essential manner that relaxes the conditions on high-order nonlinearities in [5–7], which require them to be either globally bounded, Lipschitz or linear growth.

The rest of the paper is organized as follows. Nonlinear multivariable control for known systems is introduced in Section 2. A hybrid adaptive control method for nonlinear systems with non-Lipschitz nonlinearities to simultaneously achieve stability and improved performance is proposed in Section 3. Stability and tracking errors are analyzed in Section 4. A numerical example is conducted in Section 5 to verify the effectiveness of the proposed method. Finally, Section 6 concludes this paper with some brief remarks and an outlook on future work.

2. Nonlinear multivariable control for known systems

Consider the following m-input-m-output discrete-time nonlinear system:

$$\begin{aligned} x(k+1) &= F(x(k), u(k)), \\ y(k) &= C(x(k)), \end{aligned} \tag{1}$$

where $u(k) \in \mathbb{R}^m$, $y(k) \in \mathbb{R}^m$, $x(k) \in \mathbb{R}^n$, and F and C are vector-valued nonlinear functions such that the origin is an equilibrium state.

System (1) can be linearized in the neighborhood of the origin with Taylor series expansion. Then, according to [5,6], for an observable system of order n, $x(k)$ can be expressed as the function of $y(k), \dots, y(k-n+1), u(k), \dots, u(k-n+1)$, so that the nonlinear system (1), can also be simply represented as

$$\begin{aligned} A(z^{-1})y(k+d) = B(z^{-1})u(k) + f[y(k+d-1, \dots, \\ y(k+d-n), u(k), \dots, u(k-n+1))], \end{aligned} \tag{2}$$

where $A(z^{-1})$ and $B(z^{-1})$ are two $m \times m$ matrix polynomials in the backward shift operator z^{-1} with the orders n and $n-1$, respectively; d is the relative degree; $f[\cdot] \in \mathbb{R}^m$ is a vector-valued higher-order nonlinear function [1]. Similar to [6], we need the following assumptions:

Assumption 1. The system order n and the relative degree d are known a priori.

Assumption 2. The linear parameter matrices forming $A(z^{-1})$, $B(z^{-1})$ lie in a compact region σ, and $B(0)$ is nonsingular.

Assumption 3. The system has globally uniformly asymptotically stable zero dynamics.

Remark 1. The system order and the relative degree of system (2) can be unknown and determined by the method in [17].

Remark 2. As pointed out in [5], Assumption 3 ensures that an input sequence will never grow faster than the output sequence. This assumption is a necessary condition for proving the stability of the standard adaptive control problem.

The objective of this paper is to seek a control strategy to achieve both system stability and improved performance, i.e., all the input and output signals of the closed-loop system are guaranteed to be essentially bounded, and simultaneously the system outputs optimally track the reference signals.

Introduce the following performance index [18]:

$$J_c = \|T(z^{-1})y(k+d) - R\omega(k)\|^2, \tag{3}$$

where $\omega(k) \in \mathbb{R}^m$ is a known bounded reference input vector, $T(z^{-1})$ is a stable $m \times m$ diagonal matrix weighing polynomial, and R is an $m \times m$ diagonal matrix.

The optimal control law that minimizes (3) subject to the constraint (2) is

$$H(z^{-1})u(k) + G(z^{-1})y(k) + \upsilon[\cdot] = R\omega(k), \tag{4}$$

where $H(z^{-1}) = L(z^{-1})B(z^{-1}) := H_0 + H_1 z^{-1} + \cdots + H_{n+d-2} z^{-n-d+2}$, in which $L(z^{-1})$ is an $n \times n$ matrix polynomial with the order $d-1$, $G(z^{-1}) := G_0 + G_1 z^{-1} + \cdots + G_{n-1} z^{-n+1}$ is an $n \times n$ matrix polynomial with the order $n-1$. They are uniquely determined by the equation as follows [6]:

$$T(z^{-1}) = L(z^{-1})A(z^{-1}) + z^{-d}G(z^{-1}), \tag{5}$$

and $\upsilon[\cdot] = L(z^{-1})s[\cdot]$.

When the control law (4) is used, υ is replaced by its estimation $\hat{\upsilon}[\cdot]$. Due to the uncertainty and unmodeled characteristics of υ, a universal approximator is required for this estimation. As proved in [19], ANN is an appropriate and easy-to-implement universal approximator, thus is utilized in this paper. Then the equation of the closed-loop system is

$$T(z^{-1})y(k+d) = R\omega(k) + \upsilon[\cdot] - \hat{\upsilon}[\cdot], \tag{6}$$

If the linear parts of the system are known a priori, by utilizing (6), and choosing the diagonal matrix $T(z^{-1})$ such that the determinant of $T(z^{-1})$ characterizes the poles of the closed-loop system, the tracking error of the closed-loop system is $\upsilon[\cdot] - \hat{\upsilon}[\cdot]$ with $R = T(1)$. In addition, the random search algorithm in [20] is used to calibrate the hyper-parameters based on the performance on a validation set. With well-tuned hyper-parameters and appropriate training algorithm, the estimation of ideal parameter matrix, $\hat{\boldsymbol{W}}(k)$ containing weights and biases can be obtained. Then, by taking $\hat{\boldsymbol{W}}(k)$ and $\boldsymbol{\Psi}(k)$ as the input vectors of the ANN function, the error between the estimation of unmodeled dynamics and its real value $\|\upsilon[\cdot] - \hat{\upsilon}[\cdot]\|$ can be ensured less than any specified positive number over a compact set, so that the tracking error can be as small as possible [21].

If the nonlinearity $\upsilon[\cdot]$ is small, the control law (4) can be approximated by the linear control law below:

$$H(z^{-1})u(k) + G(z^{-1})y(k) = R\omega(k). \tag{7}$$

To achieve both stability and the improved performance it is required to design a switching strategy, which is what the next section will describe.

3. Hybrid adaptive control

In this section, we first introduce a Riemann sum operator with variable step size to characterize the non-Lipschitz nonlinearities. Then linear and nonlinear estimate models are constructed for designing linear and nonlinear controllers, respectively. Finally, a switching mechanism is proposed to coordinate the two controllers to ensure both the stability and the better performance.

If the linearization parameters of the system are unknown or slowly time-varying, we use the direct adaptive control scheme. From (2) and (5), we obtain the following model:

$$T(z^{-1})y(k+d) = G(z^{-1})y(k) + H(z^{-1})u(k) + \upsilon[\bar{X}(k)], \tag{8}$$

$$\phi(k+d) = \theta^T X(k) + \upsilon[\bar{X}(k)], \tag{9}$$

where $\phi(k+d) = T(z^{-1})y(k+d)$, $\theta = [G_0, \ldots, G_{n-1}, H_0, \ldots, H_{n+d-2}]^T$, $X(k) = [y(k)^T, \ldots, y(k-n+1)^T, u(k)^T, \ldots, u(k-n-d+2)^T]^T$, and $\bar{X}(k) = [y(k), \ldots, y(k-n+1), u(k), \ldots, u(k-n-d+2)]$.

Remark 3. From Assumption 2, the parameter matrix θ lies in a certain compact region σ.

Define a Riemann sum operator p with variable step size as

$$p\alpha(k) = \sum_{i=0}^{k} \alpha(i)\Delta_i, \tag{10}$$

where α represents ϕ, X, or $\upsilon[\bar{X}]$, and $\Delta_i = 1/\max\{\phi(i+d), X(i), \upsilon[\bar{X}(i)]\}$ is variant for each time-step. Applying the summation operation p to ϕ, X, or $\upsilon[\bar{X}]$ in (9), respectively, we have

$$Y(k+d) = \theta^T\psi(k) + p\upsilon[\bar{X}(k)], \tag{11}$$

where $Y(k+d) = p\phi(k+d)$, and $\psi(k) = pX(k)$.

By taking advantage of (10), it can be obtained that the higher-order nonlinear term $\upsilon[\cdot]$ can be a non-Lipschitz nonlinearity, meanwhile its summation is always bounded by some positive constant M, i.e.

$$\|p\upsilon[\cdot]\| \leqslant M. \tag{12}$$

Remark 4. The introduction of sum operator p relaxes the condition in [5–7], which confine the nonlinearity either to be globally bounded or Lipschitz. With the proposed sum operator, when an unbounded perturbation $\upsilon[X(i)]$ occurs at time i, to name impulse function as an example, Δ_i will be zero, thus *filtering* the unbounded perturbation in the summation system automatically. Then, the controller will be designed based on the summation system instead of the original one, which will be elaborated in the next section. In this way, although it cannot compensate the unbounded perturbations, the *essential* boundedness will be achieved, i.e., all the signals are bounded almost everywhere except for the finite points that perturbation is unbounded.

It is worth noting that, essential boundedness is more practical in real-world engineering. Fully compensating an unbounded perturbation intrinsically requires an unbounded control signal which is unrealizable for any actuator and unnecessary in practice. It is more important for a controller to make systems survive the unbounded perturbation while guaranteeing the performance afterwards. For instance, in microgrid control, the center controller uses field measurement as output feedback which is usually obtained by micro phasor measurement unit (PMU) and transferred through wireless [22]. When observing the real data, one can always find the feedback containing many non-Lipschitz and very large perturbations. These perturbations are not desired to be compensated but to be eliminated. From this viewpoint, our proposed sum operation with variable step size can be considered as a special class of filter.

The proposed operator can be considered as a class of summation filter that guarantees bounded input and bounded output (BIBO) stability. It is different from the filter usually employed in parameter identification algorithms such as that in [2] from the structure design and stability property.

In this paper, two estimate models of (9) are constructed. The first one is a linear estimate model, which is defined as

$$\hat{Y}_1(k+d) = \hat{\theta}_1(k)^T\psi(k), \tag{13}$$

where $\hat{\theta}_1(k)$ is an estimate of θ at time instant k, and is updated by

$$\hat{\theta}_1(k) = \mathrm{proj}\{\hat{\theta}_1'(k)\}, \tag{14}$$

$$\hat{\theta}_1'(k) = \hat{\theta}_1(k-d) + \frac{a_1(k)\psi(k-d)e_1(k)^T}{1 + \psi(k-d)^T\psi(k-d)}, \tag{15}$$

$$a_1(k) = \begin{cases} 1 & \text{if } \|e_1(k)\| > 2M, \\ 0 & \text{otherwise}, \end{cases} \tag{16}$$

where $e_1(k)$ is the linear model error, i.e.,

$$e_1(k) = Y_1(k) - \hat{\theta}_1(k-d)^T\psi(k-d), \tag{17}$$

$\hat{\theta}_1'(k) = [\hat{G}_{1,0}(k), \ldots, \hat{G}_{1,n-1}(k), \hat{H}_{1,0}'(k), \ldots, \hat{H}_{1,n+d-2}(k)]^T$, and *proj*$\{\cdot\}$ is a projection operator satisfying

$$\mathrm{proj}\{\hat{\theta}_1'(k)\} = \begin{cases} \hat{\theta}_1'(k) & \text{if } \|\hat{H}_{1,0}(k)\| \geqslant h_{min}, \\ [\ldots, h_{min}, \ldots]^T & \text{otherwise}, \end{cases} \tag{18}$$

where h_{min} is defined by prior knowledge satisfying $h_{min} > 0$. The purpose is to refrain the control signal from being too big due to the too small identification parameter $\hat{H}_{1,0}(k)$.

The second estimate model is a nonlinear one defined as

$$\hat{Y}_2(k+d) = \hat{\theta}_2(k)^T \psi(k) + p\hat{\upsilon}^*[\bar{X}(k)], \tag{19}$$

where $\hat{\theta}_2(k)$ is another estimate of θ at time instant k, and $p\hat{\upsilon}^*[\bar{X}(k)]$ is a neural networks estimation of $p\upsilon^*[\bar{X}(k)]$ at time instant k with $\upsilon^*[\bar{X}(k)] = Y(k+d) - \hat{\theta}_2(k)^T\psi(k)$, i.e.,

$$p\hat{\upsilon}^*[\bar{X}(k)] = \mathrm{NN}[\hat{W}(k), \psi(k)], \tag{20}$$

where NN[·] represents the structure of the adopted neural networks, $\psi(k)$ is the input vector, and $\hat{W}(k)$ is the estimate of the ideal weight matrix W^*.

Similar to [5], there is no restriction on how the parameters $\hat{\theta}_2(k)$ and $\hat{W}(k)$ are updated except that they always lie inside some predefined compact region Ω, i.e.,

$$\hat{\theta}_2(k), \hat{W}(k) \in \Omega \qquad \forall k. \tag{21}$$

The nonlinear model error is

$$e_2(k) = Y_2(k) - \hat{\theta}_2(k-d)^T\psi(k-d) - p\hat{\upsilon}^*[\bar{X}(k-d)]. \tag{22}$$

By the linear estimate model (13), we have the linear adaptive controller C_1 for summation system:

$$\hat{\theta}_1(k)^T\psi(k) = pR\omega(k). \tag{23}$$

From the nonlinear estimate model (19), we obtain the neural-networks-based nonlinear adaptive controller C_2 for summation system:

$$\hat{\theta}_2(k)^T\psi(k) + p\hat{\upsilon}^*[\bar{X}(k)] = pR\omega(k). \tag{24}$$

To compute the controller for original system, we conduct the following transformation:

$$u(k) = S \times (\psi(k) - \psi(k-1))/\mathrm{sat}(\Delta_k). \tag{25}$$

where

$$S = \begin{bmatrix} \mathbf{0}_{1\times n-1} & 1 & \mathbf{0}_{1\times n+d-2}, \end{bmatrix}$$

$$\mathrm{sat}(\Delta_k) = \begin{cases} \Delta_k & \text{if } \|\Delta_k\| > \Delta_{\min}, \\ \Delta_{\min} & \text{otherwise.} \end{cases}$$

The saturation function is designed to prohibit $u(k)$ tending to zero when unbounded perturbation occurs. Consequently, the unbounded perturbation is not fully compensated at time k. However, when the perturbation returns to the normal operational range, the controller (25) is still capable of coping with it instead of diverging afterwards.

In the following, hybrid adaptive control between the linear adaptive controller C_1 and the neural networks nonlinear adaptive controller C_2 is given. The structure of the hybrid system is illustrated in Fig. 1, where $j = 1$ and 2. $j = 1$ denotes linear, and $j = 2$ denotes nonlinear. A similar switching rule as in [5] is proposed below

$$J_j(k) = \sum_{l=d}^{k} \frac{a_j(l)(\|e_j(l)\|^2 - 4M^2)}{2(1+\psi(l-d)^T\psi(l-d))} + c\sum_{l=k-N+1}^{k}(1-a_j(l))\|e_j(l)\|^2, \tag{26}$$

$$a_j(k) = \begin{cases} 1 & \text{if } \|e_j(k)\| > 2M, \\ 0 & \text{otherwise,} \end{cases} \tag{27}$$

where N is an integer and $c \geqslant 0$ is a predefined constant.

By comparing $J_1(k)$ and $J_2(k)$, the adaptive controller C_* corresponding to the smaller $J_*(k)$ is chosen to control the system. Note that the performance index (26) is composed of two parts. The first part, $\sum_{l=d}^{k} a_j(l)(\|e_j(l)\|^2 - 4M^2)/2(1+\psi(l-d)^T\psi(l-d))$, is used to distinguish between signals with different growth rates, so that boundedness of all the signals can be established. The second part, $c\sum_{l=k-N+1}^{k}(1-a_j(l))\|e_j(l)\|^2$, is a measure of the prediction error over a finite window and is included to improve performance. When the neural networks are degraded or disturbed, the e_2 increases, consequently, J_1 is less than J_2 and the controller C_1 is chosen. C_1 keeps working to guarantee the stability until the neural networks based controller recovers. Then e_2 decreases, accordingly, J_1 is greater than J_2 and the controller C_2 is chosen to improve performance. The parameters c and M in (26) can influence the tracking performance. Parameter c denotes the weight of tracking performance, which is usually selected around 1.5 to balance the stability and control performance [6]. As M decreases, the accuracy of linear parts will increase. However, a too small M will lead to sluggish convergence of parameter updating process.

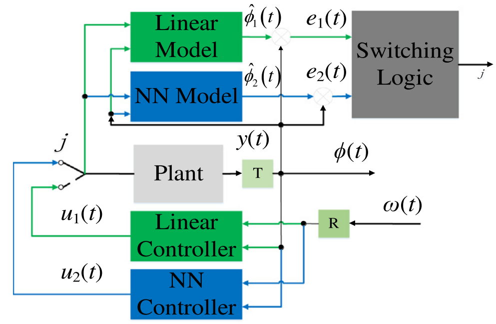

Fig. 1. The block diagram of the hybrid adaptive controller.

Remark 5. According to the theory of switched systems [23], it is possible to give bad control performance or even lead to instability by frequent switching stabilizing controllers or stable subsystems, and also it is possible to obtain better performance or guaranteed stability by frequent switching controllers or unstable subsystems. That is, a switching law can determine whether an overall switched system is stable or how the control performance will be after switching among controllers or subsystems. Therefore, constructing an appropriate switching law plays an essentially important role in how control performance of the switched system is. The switching law proposed in this paper gives better performance, i.e., the proposed hybrid adaptive control based on the multiple models theory guarantees both (BIBO) stability and improved performance of the system.

The hybrid adaptive control algorithm proposed in this paper is composed of the identification algorithm for controller parameters, the linear adaptive controller, the neural networks nonlinear adaptive controller, and the switching mechanism. It can be summarized as follows:

Step 1: Measure $y(k)$ and construct datum vector $\psi(k-d)$;
Step 2: Calculate the model errors $e_1(k)$ and $e_2(k)$ by (17) and (22), and calculate $J_1(k)$ and $J_2(k)$ by (26) and (27);
Step 3: Compare $J_1(k)$ and $J_2(k)$, and choose the controller C_*, (23) or (24), corresponding to the smaller $J_*(k)$;
Step 4: Estimate the controller parameter $\hat{\theta}_*(k)$ using (14)–(18) or (20)–(22), and calculate the current control input $u(t)$ from the controller C_* to be applied to system (2);
Step 5: Let $k = k+1$, and return to Step 1.

4. Analysis of stability and tracking errors

In this section, the theoretical results on stability and tracking errors are first stated, and then the rigorous proof is provided.

Theorem 1. *For system (2), the hybrid adaptive control algorithm (13)–(27) ensures that the input and output signals of the closed-loop system are essentially bounded, i.e., bounded almost everywhere except for the finite points that $\upsilon[\cdot]$ is unbounded.*

In addition, by properly choosing the structure and hyper-parameters of the ANN, for a predefined arbitrarily small positive number ε, the tracking error of the closed-loop system satisfies

$$\lim_{k\to\infty}\|\bar{e}'(k)\| = \lim_{k\to\infty}\|T(z^{-1})y(k) - R\omega(k-d)\| < \varepsilon.$$

almost everywhere except for the points that $\upsilon[\cdot]$ is unbounded.

Proof. First, define $\tilde{\theta}_1(k) = \hat{\theta}_1(k) - \theta$, and then by (15), we have

$$\tilde{\theta}_1(k) = \tilde{\theta}_1(k-d) + \frac{a_1(k)\psi(k-d)e_1(k)^T}{1+\psi(k-d)^T\psi(k-d)}.$$

Bearing the similarity in [5], it follows that

$$\|\tilde{\theta}_1(k)\|^2 \leqslant \|\tilde{\theta}_1(k-d)\|^2 - \frac{a_1(k)(\|e_1(k)\|^2 - 4M^2)}{2(1+\psi(k-d)^T\psi(k-d))}. \tag{28}$$

Since $a_1(k) = 1$ for $\|e_1(k)\| > 2M$, and is 0 otherwise, $\{\|\tilde{\theta}_1(dk)\|^2\}$ is d nonincreasing sequences. Hence, $\hat{\theta}_1(k)$ is bounded. Moreover,

$$\lim_{N\to\infty}\sum_{k=d}^{N}\frac{a_1(k)(\|e_1(k)\|^2 - 4M^2)}{2(1+\psi(k-d)^T\psi(k-d))} < \infty, \tag{29}$$

$$\lim_{k\to\infty}\frac{a_1(k)(\|e_1(k)\|^2 - 4M^2)}{2(1+\psi(k-d)^T\psi(k-d))} \to 0. \tag{30}$$

From (17) and (23), we have

$$\begin{aligned} e_1(k) &= Y(k) - \hat{\theta}_1(k-d)^T\psi(k-d) \\ &= pT(z^{-1})y(k) - pR\omega(k-d). \end{aligned} \tag{31}$$

Note that $e_1(k)$ is a summation, which consists of a series of identification errors at each instant, i.e., $e_1(k) = pe'_1(k)$, where $e'_1(k) = T(z^{-1})y(k) - R\omega(k-d)$.

By (31) and the stability of $T(z^{-1})$, along with Assumption 3, there exist positive constants c_1, c_2, c_3, and c_4 such that

$$|y_i(k)| \leqslant c_1 + c_2 \max_{0\leqslant\tau\leqslant k;1\leqslant i\leqslant n}|e'_{1i}(\tau)| \qquad i = 1,2,\ldots,n,$$

and

$$|u_i(k-d)| \leqslant c_3 + c_4 \max_{0\leqslant\tau\leqslant k;1\leqslant i\leqslant n}|y_i(\tau)| \qquad i = 1,2,\ldots,n.$$

Since

$$\begin{aligned} X(k-d) = [&y(k-d)^T,\ldots,y(k-n-d+1)^T, \\ &u(k-d)^T,\ldots,u(k-n-2d+1)^T]^T, \end{aligned}$$

it follows that there exist positive constants c_5, c_6, c_7 and c_8 such that

$$\|X(k-d)\| \leqslant c_5 + c_6 \max_{0\leqslant\tau\leqslant k}\|e'_1(\tau)\|. \tag{32}$$

$$\|\psi(k-d)\| \leqslant c_7 + c_8 \max_{0\leqslant\tau\leqslant k}\|e_1(\tau)\|. \tag{33}$$

From (32), the boundedness of the input and output signals are determined by the boundedness of $e'_1(k)$. To prove the boundedness of $e'_1(k)$, we can prove the essential boundedness of $e_1(k)$ instead.

Now assume that $e_1(t)$ is unbounded. Then by (16), for a sufficiently large positive constant L, when $k > L$, we have $\|e_1(k)\| > 2M$ and $a_1(k) = 1$. This indicates that the numerator in (30) is a positive real scalar sequence, and thus there exists a monotony increasing sequence $\|e_1(k_n)\|$, such that $\lim_{k_n\to\infty}\|e_1(k_n)\| = \infty$.

However, we can derive that

$$\begin{aligned} &\lim_{k_n\to\infty}\frac{a_1(k_n)(\|e_1(k_n)\|^2 - 4M^2)}{2(1+\psi(k_n-d)^T\psi(k_n-d))} \\ &= \lim_{k_n\to\infty}\frac{a_1(k_n)(\|e_1(k_n)\|^2 - 4M^2)}{2(1+\|\psi(k_n-d)\|^2)} \\ &\geqslant \lim_{k_n\to\infty}\frac{a_1(k_n)(\|e_1(k_n)\|^2 - 4M^2)}{2(1+(c_7+c_8\,max_{0\leqslant\tau\leqslant k}\|e_1(\tau)\|)^2)} \\ &\geqslant \lim_{k_n\to\infty}\frac{a_1(k_n)(\|e_1(k_n)\|^2 - 4M^2)}{2(1+(c_7+c_8\|e_1(k_n)\|)^2)} \\ &\geqslant \frac{1}{2C_8^2} \\ &> 0, \end{aligned}$$

which contradicts (30), and hence the assumption that $e_1(k)$ is unbounded is false, i.e., $\psi(k-d)$ is bounded. Thus, by the definition (10), the input $u(k-d)$ is bounded and output $y(k)$ is bounded almost everywhere except for the points that υ is unbounded. Consequently, $X(k-d)$ is essentially bounded when the linear adaptive controller is used alone.

Second, by (22) and (24), we have

$$\begin{aligned} e_2(k) &= Y(k) - \hat{\theta}_2(k-d)^T\psi(k-d) - p\hat{\upsilon}^*[\bar{X}(k-d)] \\ &= pT(z^{-1})y(k) - pR\omega(k-d). \end{aligned} \tag{34}$$

By (34) and the stability of $T(z^{-1})$, along with Assumption 3, there exist positive constants d_1, d_2, d_3, and d_4 such that

$$|y_i(k)| \leqslant d_1 + d_2 \max_{0\leqslant\tau\leqslant k;1\leqslant i\leqslant n}|e_{2i}'(\tau)| \qquad i = 1,2,\ldots,n,$$

$$|u_i(k-d)| \leqslant d_3 + d_4 \max_{0\leqslant\tau\leqslant k;1\leq i\leqslant n}|y_i(\tau)| \quad i = 1,2,\ldots,n.$$

Therefore, similarly, there exist positive constants d_5, d_6, d_7 and d_8 such that

$$\|X(k-d)\| \leqslant d_5 + d_6 \max_{0\leqslant\tau\leqslant k}\|e'_2(\tau)\|. \tag{35}$$

$$\|\psi(k-d)\| \leqslant d_7 + d_8 \max_{0\leqslant\tau\leqslant k}\|e_2(\tau)\|. \tag{36}$$

By (27), the second term in (26) is always bounded, so $J_1(k)$ is bounded by employing (29). For $J_2(k)$, there are two possibilities:

Case 1: $J_2(k)$ is bounded.

By the switching rule (26), it follows that

$$\lim_{k\to\infty}\frac{a_2(k)(\|e_2(k)\|^2 - 4M^2)}{2(1+\psi(k-d)^T\psi(k-d))} \to 0.$$

Therefore, the model error of the closed-loop system, $e(k) = e_1(k)$ *or* $e_2(k)$, satisfies

$$\lim_{k\to\infty}\frac{a(k)(\|e(k)\|^2 - 4M^2)}{2(1+\psi(k-d)^T\psi(k-d))} \to 0, \tag{37}$$

where

$$a(k) = \begin{cases} 1 & if\,\|e(k)\| > 2M, \\ 0 & otherwise. \end{cases} \tag{38}$$

Case 2: $J_2(k)$ is unbounded.

Since $J_1(k)$ is bounded, there exists a constant k_0 such that $J_1(k) \leqslant J_2(k), \forall k \geqslant k_0$. Therefore, when $k \geqslant k_0+1$, by the switching mechanism, the model $e(k) = e_1(k)$, and also satisfies (37).

From (33), (36) and (37), using the same reasoning as above, it follows that $\psi(k-d)$ is bounded, and thus the input and output signals of the closed-loop hybrid system are essentially bounded.

Finally, from (37) and the boundedness of $\psi(k-d)$, the model error $e_j(k)$, $j = 1, 2$, satisfies

$$\lim_{k\to\infty}\|e_j(k)\| \leqslant 2M. \tag{39}$$

Then by the switching rule (26) and (27), when $k \to \infty$, the system chooses the controller corresponding to the smaller model error as the control input of the switching system. So from (31) and (34), the tracking error of the summation system, $\bar{e}(k) = Y(k) - pR\omega(k-d)$, is equivalent to the smaller model error.

We now will prove that when $k \to \infty$, the tracking error is equivalent to the nonlinear model error, i.e., the nonlinear model error can always be less than the linear model error. For the nonlinear model error, from (22), we have

$$\begin{aligned} e_2(k) &= Y_2(k) - \hat{\theta}_2(k-d)^T \psi(k-d) - p\hat{\upsilon}^*[\bar{X}(k-d)] \\ &= Y_2(k) - (Y_2(k) - p\upsilon^*[\bar{X}(k-d)]) - p\hat{\upsilon}^*[\bar{X}(k-d)] \\ &= p\upsilon^*[\bar{X}(k-d)] - p\hat{\upsilon}^*[\bar{X}(k-d)]. \end{aligned} \tag{40}$$

When the structure and parameters of a neural network are chosen properly, for a predefined arbitrary small positive number $\varepsilon(< \lim_{k\to\infty} \|e_1(k)\|)$, the tracking error $\|p\upsilon^*[\bar{X}(k-d)] - p\hat{\upsilon}^*[\bar{X}(k-d)]\| < \varepsilon$ can be achieved. Thus, when $k \to \infty$, the nonlinear model error can be less than the linear model error, consequently, the tracking error of the system will be $e_2(k)$, which satisfies

$$\lim_{k\to\infty} \|\bar{e}(k)\| = \lim_{k\to\infty} \|e_2(k)\| < \varepsilon. \tag{41}$$

It is obvious that for the points that $\upsilon[\cdot]$ is bounded, the above conclusion also holds for tracking error $\bar{e}'(k)$, i.e.,

$$\lim_{k\to\infty} \|\bar{e}'(k)\| = \lim_{k\to\infty} \|T(z^{-1})y(k) - R\omega(k-d)\| < \varepsilon. \quad \square$$

5. Simulation studies

In this section, a numerical nonlinear system is presented to illustrate the effectiveness of the proposed method by comparing with the one in [5].

Consider the following double-input-double-output discrete-time nonlinear dynamical system in the form of system (2):

$$\begin{aligned} y_1(k+1) &= 0.6y_1(k) + 1.2y_2(k) + 1.5y_1(k-1) \\ &\quad + 0.3y_2(k-1) + 1.1u_1(k) + 0.8u_2(k) \\ &\quad + 0.54u_1(k-1) + \sqrt{|u_1(k)-1|} \\ &\quad + \frac{u_1(k) + u_2(k-1) + y_1(k) + y_2(k-1)}{1 + u_1(k)^2 + u_2(k-1)^2 + y_1(k)^2 + y_2(k-1)^2} \\ y_2(k+1) &= 2.4y_1(k) + 0.1y_2(k) - 0.2y_1(k-1) \\ &\quad + 1.8y_2(k-1) + 1.25u_2(k) + 0.32u_1(k-1) \\ &\quad + 0.1u_2(k-1) + \sqrt{|u_2(k)-1|} \\ &\quad + \frac{u_1(k) + u_2(k-1) + y_1(k) + y_2(k-1)}{1 + u_2(k)^2 + u_1(k-1)^2 + y_2(k)^2 + y_1(k-1)^2}. \end{aligned}$$

The origin is an equilibrium point, the relative degree $d = 1$ and the system order $n = 2$. The sampling time interval $\Delta T = 1$ s. It can be observed that, unlike [6], the higher-order nonlinearities in the above nonlinear system are *not* Lipschitz continuous. Reference trajectories, similar to [6], $w_1 = 1.5(\sin 2t/10 + \sin 2t/25)$ and $w_2 = w_1$ are chosen to be followed. The weighting matrix polynomial

$$T(z^{-1}) = \begin{bmatrix} 1 - 0.1z^{-1} & 0 \\ 0 & 1 - 0.1z^{-1} \end{bmatrix}$$

ensures the assigned poles are both 0.1, and the weighting matrix

$$R = \begin{bmatrix} 0.9 & 0 \\ 0 & 0.9 \end{bmatrix}$$

is chosen.

A back-propagation neural network with single hidden layer and adaptive learning rate in batch mode is chosen. In order to determine the optimal number of hidden nodes, a cross-validation procedure is used, which starts by moving bottom-up [24]. As a result, when the number of the hidden nodes is larger than 6, the cross-validation error cannot be reduced. So, the number 6 is chosen as the optimal number of hidden nodes. The parameters in (27) are chosen to be $c = 1.5$ and $N = 2$. The learning rate is 0.1.

When the hybrid adaptive control algorithm proposed in this paper is implemented, Figs. 2–3 and Fig. 4 show the tracking performance and the corresponding switching signals, respectively. The root mean square errors (RMSEs) equal for $[y_1, y_2]$ is [0.7966, 1.1912]. Note that the real outputs did not track the references well in the first 10 time steps. These overshoots can be attributed to the nature of adaptive control, because it needs

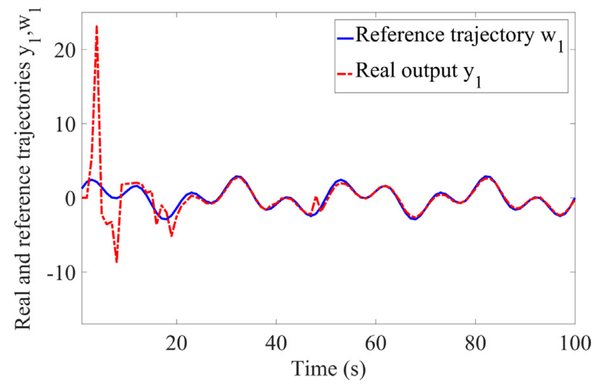

Fig. 2. The tracking performance of output y_1 using the proposed method.

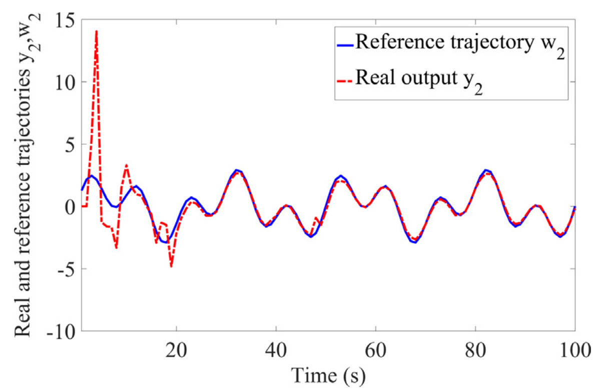

Fig. 3. The tracking performance of output y_2 using the proposed method.

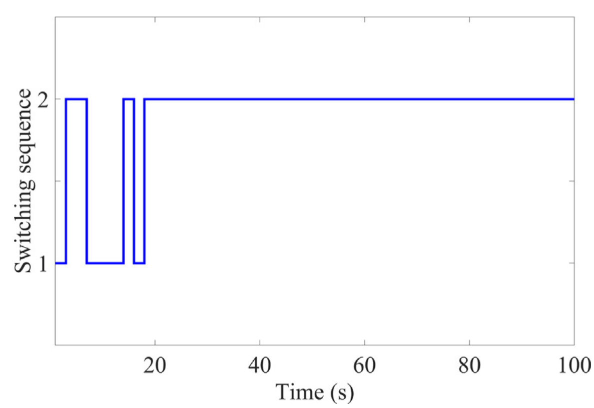

Fig. 4. The switching sequence of the proposed method.

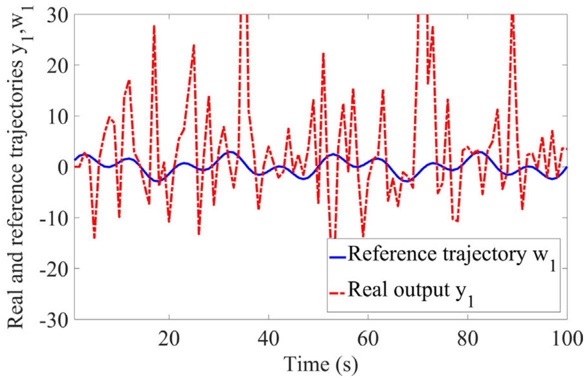

Fig. 5. The tracking performance of output y_1 using the contrastive method.

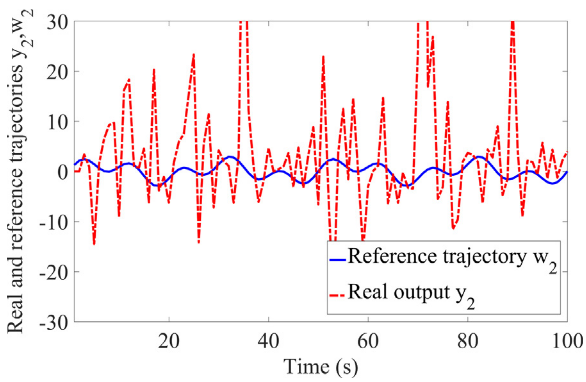

Fig. 6. The tracking performance of output y_2 using the contrastive method.

some time to update the parameters. After that good tracking performance of the output signals is achieved. The switching signal illustrates that even though the neural network nonlinear controller works very well most of the time, it degrades occasionally, and then the linear controller has to take over until the neural network controller works again. However, if the neural network controller cannot recover, the linear controller will work all the time. Different sets of parameters of c, N and the leaning rate of the neural network are also simulated, with varying tracking performances, but signals in the system are always bounded.

To further verify the effectiveness of the proposed method for non-Lipschitz nonlinearity, the method in [5] is implemented on the same simulation conditions. The tracking performance is shown in Figs. 5–6 with RMSEs equal to [48.9881, 47.5963] for y_1 and y_2, respectively. The inaccuracy indicate that the original control algorithm in [5] cannot stabilize the system with non-Lipschitz nonlinearities. As shown in Fig. 7, the neural network controller is chosen almost all the time. This is due to the strong nonlinearities in this case.

The simulation studies for this numerical example illustrate that the proposed hybrid adaptive control method can achieve both stability and improved performance simultaneously.

6. Conclusion

Based on the framework established in [5,6], a hybrid adaptive control method is proposed for a class of MIMO discrete-time nonlinear dynamic systems with non-Lipschitz nonlinearities. The non-Lipschitz nonlinearities were compensated by introducing a new sum operator, which is a distinguishing feature compared to existing methods in the literature. It is shown that properly switching between the linear controller and the nonlinear one achieves both bounded input and output signals of

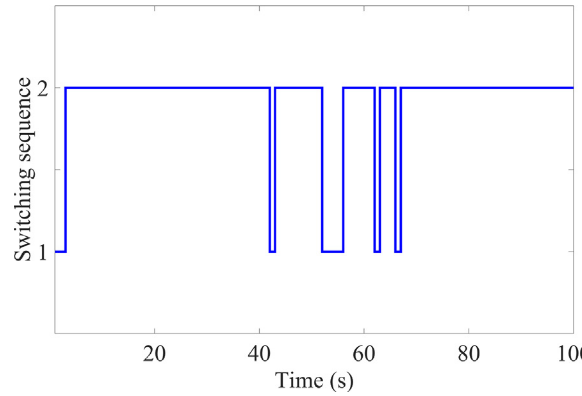

Fig. 7. The switching sequence of the contrastive method.

the closed-loop hybrid switching system and improved tracking performance subject to mild assumptions.

Extension on the proposed method may lie in the following aspects. The current method requires minimum-phase assumption on the considered systems, but many practical systems are essentially non-minimum-phase [25], which is also a difficult problem in nonlinear control [26]. Therefore, it is meaningful but not straightforward task to extend this method to non-minimum-phase case, which is our ongoing work. The method herein uses the neural networks to approximate the higher-order nonlinearities, where the learning problem is a heuristic optimization problem. Thus the global optimality cannot be guaranteed [27]. It is suggested to first apply new non-heuristic approximation methods [27] to optimally approximate the higher-order nonlinearities and then adopt the deterministic global optimization techniques [27,28] to solve the optimization problems to global optimality. The future work will focus on applying the proposed method to the pulp neutralization process in [29].

CRediT authorship contribution statement

Jun Fu: Conceptualization, Methodology, Writing – review & editing, Supervision. **Zixiao Ma:** Methodology, Software, Validation, Writing – original draft. **Yue Fu:** Methodology, Writing – review & editing. **Tianyou Chai:** Supervision.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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